ENTROPY AND CONDITIONED NEGENTROPY – MATHEMATICAL MODELS FOR THE ANALYSIS OF QUALITY

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ABSTRACT

The subject of entropy and its application of the management of quality has been approached by other authors as well [1], [2], [3], [4]. Through our transdisciplinary approach, we would like to contribute to the development of this subject. To extend the range of mathematical models applied for the scientific approach of quality and quality management, and simulated with programs such as Microsoft Excel, we brought a contribution by presenting our own results, as their authors or co-authors, published in specialty journals and works presented at conferences, among which: we defined quality entropy, we simulated it in Microsoft Excel, and we applied it for the analysis of key programs of study [5], [6]; we defined conditioned negentropy [7], and we applied it [8], [9]. This paper adds new results that use the concepts of entropy, conditioned negentropy of a tangible or intangible product’s quality. The research methods used in the realization of this paper are the bibliographical research method and the creation of new models. The main conclusions are that mathematical models of entropy and negentropy is compatible with the epistemological simplicity required in the field of quality and quality management.

KEYWORDS: entropy, conditioned negentropy, quality

1. Introduction

What are the mathematical models used for the approach of quality and quality management so far?

For the scientific approach of quality and its management have been applied so far mathematical models related to:
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<th>Field</th>
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<tr>
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<td>• the regression theory as a prediction instrument in quality management [13],</td>
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<td>Probability theory</td>
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<td>Multicriterial or multiobjective mathematical programming</td>
<td>• optimizing product quality [21], [22], [23], [24], [25], [26]</td>
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<td>Fuzzy systems</td>
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<td>Time series</td>
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<td>• the quality loss function [15], [31]</td>
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To extend the range of mathematical models applied for the scientific approach of quality and quality management, and simulated with programs such as Microsoft Excel, we brought a contribution by presenting our own results, as their authors or co-authors, published in specialty journals and works presented at conferences, among which: we defined quality entropy, we simulated it in Microsoft Excel, and we applied it for the analysis of key programs of study [5], [6]; we defined conditioned negentropy [7], and we applied it [7], [8].

Mathematical models present a few defining elements by ensuring *sine qua non* conditions. The mathematical models are: 

artifacts, intellectual constructions of the human mind, methods of testing hypotheses that may be algorithmized and simulated, the accuracy of the models is only relative and not absolute, less complex than the “real modeled decoupling”, perfectible and sustainable (as they do not wear out over time).

2. Entropy – Mathematical Model for the Analysis of Quality

The term comes from Greek, where “entropy” means “change” [32]. It is generally agreed that the history of its evolution has the following landmarks [32]:

Year 1865. German physician and mathematician Rudolf Julius Emanuel Clausius
(1822-1888), founder of thermodynamics, introduced the concept of entropy in order to measure how close a thermodynamic system is to thermodynamic equilibrium.

The Greek mathematician Constantin Carathéodory (1873-1950) turned the concept into a formula:

\[ S_A = \int_{A_0}^{A} \frac{dq_{rew}}{T} \]

where:
- \( dq_{rew} \) represents the amount of heat exchanged with the exterior in a reversible transformation,
- \( A \) is the state to which the empirical entropy \( S_A \) refers,
- \( A_0 \) is the reference state,
- \( T \) is the absolute temperature at which the transformation takes place,
- \( S \) state function.

**Year 1872.** Austrian mathematician and physician Ludwig Boltzmann (1844-1906) introduced the concept of mechanical entropy:

\[ S = k \log W, \]

where:
- \( S \) represents the entropy of an ideal gas,
- \( k \) Boltzmann constant,
- \( W \) quantity that exists with a certain probability (Wahrscheinlichkeit).

**Year 1900.** German physician Max Karl Ernst Ludwig Planck (1858-1947) turned formula (2) that is still used nowadays:

\[ S = -k \sum_{x=1}^{x=n} p_x \log p_x. \]

**Year 1876.** Physician, chemist, and mathematician J. Willard Gibbs (1839-1903) introduced the Gibbs entropy:

\[ dS = \frac{\delta Q}{T}. \]

**Year 1927.** American mathematician John von Neumann (1903-1957) extended the Gibbs entropy to Neumann entropy:

\[ S = -tr(\rho \log \rho). \]

where
- \( \rho \) = density matrix, it is the mathematical tool of quantum statistical mechanics.

**Year 1948. Shannon entropy.** American mathematician and electronic engineer Claude Elwood Shannon (1916-2001) gave the concept of entropy a probabilistic interpretation. The classic axiomatic constructions of the Shannon entropy are made through the Shannon – Faddeev and Shannon – Hincin axioms [33].

If \( X \) is a random discreet variable with values \( x_k, k = 1, ..., n \), taken with probabilities \( p_k \), according to Shannon’s formula, the entropy is:

\[ H(X) = -\sum_{x=1}^{x=n} p_x \log p_x. \]

where:
- \( \log p_x \) is a logarithm to a base higher than 1 that is applied to value \( p_x \) [34], [35].

If \( X \) is a random continuous variable, then [34]:

\[ H(X) = -\int f(x) \log f(x). \]

Among the properties of the Shannon entropy, the following have been demonstrated: non-negativity, continuity, symmetry, expansion, recursivity, super-additivity, maximality, uniform distribution, independency, concavity, boundedness etc.

**Classic entropies derived from the Shannon entropy** are: continuous entropy, relative entropy, entropy of a composite system, entropic redundancy, and conditioned entropy.

In the information theory have been introduced, through the years, over 30 years entropy measures that generalize Shannon’s entropy, namely entropies of parametric type, of trigonometric type, weighted entropies etc.

**Other types of entropy, generalizations of Shannon’s entropy** – non-extensive entropies are: Tsallis entropy – 1988 [36], [37], Kaniadakis entropies – 1996 [38], Sharma-Taneja-Mittal – 1975 [39], [40], and Abe – 1977 [41]. They are expressed by generalized logarithmic functions [20]:
Out of those mentioned, the Tsallis entropy is the most used. There are two sets of axioms of the Tsallis entropy, obtained by Furuichi and Suyari by extending the Shannon entropy.

**Year 1988.** Brazilian physician of Greek origin Constantin Tsallis (b. 1943) extended the Shannon entropy to a parametric form named *Tsallis entropy*, which is characterized by concavity, non-additivity, and convergence towards the classic Shannon entropy. In physics, *Tsallis entropy* is a generalization of the Boltzmann-Gibbs entropy [32], [36]:

\[ H = - \sum x p(x) \log p(x), \quad p(x) \log p(x) = 0 \text{ for } x = 0. \]

(8)

Some extensions of formula (8), introduced by Rényi [42], Aczel-Daróczzi [43], Kapur [44], Havrda-Charvát [45], [46], Arimoto [47], Sharma-Mittal [48], Taneja [39], [40], [49], Ferreri [50], Belis-Guiaşu [51], Varma [52], Gil [53], Picard [18], Santanna-Taneja [54], Sheraz [55], [56] and others, are of the form:

\[ S_q(p) = \frac{k}{q-1} \left(1 - \sum p_k^q\right), \]

where

\[ q \text{ is a real parameter for which, if } q \to 1, \text{ the Tsallis entropy is reduced to Boltzmann-Gibbs entropy:} \]

(9)

\[ S_q[p] = \frac{1}{q-1} \left(1 - \int p^q(x) dx\right). \]

(10)

Entropy has numerous applications in several fields, as well as in methods of assessing the importance coefficients of criteria from multicriterial decision-making methods.

Other examples of its applications are:

- *entropy in nature* – used for the analysis of species diversity [57];
- *economic entropy* – it expresses the degradation process of material and energy resources, process that takes places both through the socio-economic activity, as well as outside of it [41];
- *informational entropy* – in economics and finance: for investigating the distribution of expenses on categories of consumption, with respect to income, the consumers’ preferences, the evolution in the structure of sales of goods and services etc.
- *financial entropy* – Kaniadakis, Renyi, Shafee, and Ubriaco entropies are correlated with Garch models of volatility [55];
- *empirical entropy* – applications in thermodynamics, mechanics etc. [58];
- *entropy in society* – Shannon entropy and weighted entropies are used in the analysis of cultural, linguistic, and human resource diversity in organizations [57];
- *quality entropy* – for improving the quality of tangible and intangible assets [5], [6];
- *negative entropy/negentropy* – It offers explanations regarding “order inside living organisms”. It is used in measuring the distance to normality, and in risk management, where it is understood as force that follows an efficient organizational behavior, and it leads to a state of predictable equilibrium. With origins in thermodynamics, negentropy is used in the systemic approach as being synonymous to the cohesion force. American mathematician Norbert Wiener (1894-1964) described negentropy as being the physical translation of information [35].

By analogy, by using Shannon entropy, we defined the *quality entropy at a t time interval, for a product, service of process as the degree of unknown regarding the accomplishment of its quality characteristics at a t given time.*

If the product or service is subject to analysis at a given time from the perspective of a values n quality characteristics \(x_{q1}, x_{q2}, \ldots, x_{qn}\), whereas \(p_{k1} = P(x_{qk1} > \alpha_{k1}), \ldots, p_{kn} = P(x_{qn} > \alpha_{kn})\) represent the probabilities that the values of the n quality characteristics should be higher than the values of \(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}\), and therefore
we can define the $Q'_t$, finite probability field, expressed in the formula below:

$$Q'_t: \left( \frac{x_{q_1} x_{q_2} \ldots x_{q_n}}{p_{i_1} p_{i_2} \ldots p_{i_n}} \right),$$

$$p_{i_1} + \ldots + p_{i_n} = 1.$$ 

Hence, mathematically, quality entropy at a $t$ given time, for a product, service or process, may be calculated as follows [5]:

$$H_n(Q'_t) = - \sum_{k=1}^{n} p_i \log p_i,$$

where $H_n(Q'_t)$ is quality entropy at a $t$ given time, for a product/service/process.

$\log p_i$ a logarithmic function in a superunitary base applied to the $p_i$ value.

In order to make sure that the definition is accurate, let us assume that the term be equal to zero, when $p_i = 0$. This term is defined as a contribution of the $k$ quality characteristic with the probability within the quality entropy in formula (13).

3. Conditioned Negentropy – Mathematical Model for the Analysis of Quality

The concept of negentropy was defined by Erwin Schrödinger [35] and was applied for measuring the distance to normality and in risk management, where it is seen as force that follows efficient organizational behavior and leads to a state of predictable balance.

Within this section we defined the notion of conditioned negentropy for the event $A_t$: “the event that the quality level of higher education to be the expected one at the time point $t$.”

Definition. It is called conditioned negentropy of event $A$ the value calculated with the formula:

$$NH_t = - \sum_{k=1}^{n} p_i(q_t^i) \log p_i(q_t^i),$$

where $A_{t}^{e}$ represents the event in which a certain quality characteristic is realized at a certain level on a Likert scale at a set time point $t$;

$p_{i}^{q_t}$ – the probability of event $A_{t}^{e}$ at the time point $t$;

$q_{t}^{e}$ – the probability of event $A$ conditioned by $A_{t}^{e}$ at the time point $t$.

The value defined by formula (14) represents a measure of information regarding the occurrence of event $A$, based on the data obtained about quality characteristics and the correlations between these characteristics.

Property. The conditioned negentropy $NH_t$ from formula (14) can be approximated with

$$\sum_{k=1}^{n} p_i q_t^i (2 - p_i + q_t^e).$$

This approximation results from the development formula of $\ln x$ in a power series, around the point $x=1$ and by substituting $\ln x$ with the first term of the series.

Observation. The negentropy defined in formula (14) is always a positive value, which emphasizes the fact that we always have information about the event $A$ and its quality characteristics.

Let the following be a random variable at the time point $t$,

$$N_{H_t} = \left( \frac{x_{q_1} x_{q_2} \ldots x_{q_n}}{p_{i_1} p_{i_2} \ldots p_{i_n}} \right).$$

Considering each value from the second row of formula (15) as an element $P_i$ on the real axis, $P_i = p_i q_t^i$, we may define a neighborhood of the form

$$\left[ P_i^{(0)}, P_i^{(2)} \right] = L_i, \quad \text{for} \quad t = 1, n,$$

with the center in the point $P_i^{(0)} = p_i q_t^0$ on the real axis.

Thus, the random variable associated to conditioned negentropy, meaning formula (15), turns into a variable in which the values $P_i = p_i q_t^i$ may be chosen from a number of possible sets of values:

$$NH_t:\left( \frac{x_{q_1} x_{q_2} \ldots x_{q_n}}{p_{i_1} p_{i_2} \ldots p_{i_n}} \right) = \left[ P_i^{(0)}, P_i^{(2)} \right] = \left[ p_{i_1}^{(0)}, p_{i_n}^{(0)} \right].$$
We choose a “scatter” radius of a real number $x = x_{\text{scatter}}$, which we denote by $r(x)$, with the formula: $r(x) = \frac{1}{\alpha x}$, for any positive $x$. Thus, in formula (16) will be:

$$P_i^{(1)} = P_i - r(x_{\text{scatter}}),$$

$$P_i^{(2)} = P_i + r(x_{\text{scatter}}),$$

$$[P_i - r(x_{\text{scatter}}), P_i + r(x_{\text{scatter}})] = I_i.$$

And the model of conditioned negentropy will be transposed into the fuzzy theory, as it will not use probabilities anymore, but possibilities that the values $P_i = P_i^{(1)}$ may take.

4. Conclusions

In future research, specialists, practitioners, as well as anyone interested are invited to perfect the quantitative methods of management and their applications in accordance with the fundamental changes in the contemporary world, and in accordance with the evolution of software in the field.

The topics approached in this paper represent a direction of study for the author and will be analyzed and presented in more detail in the monograph “Contributions to the scientific approach of quality and quality management through modeling and simulation” (in Romanian).

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REFERENCES


